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The Strong Chebyshev Distribution and Orthogonal Laurent Polynomials

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The strong Chebyshev distribution and the Chebyshev orthogonal Laurent polynomials are examined in detail. Explicit formulas are derived for the orthogonal Laurent polynomials, uniform convergence of the associated continued fraction is established, and the zeros of the Chebyshev L-polynomials are given. This provides another well-developed example of a sequence of orthogonal L-polynomials. © 1998 Academic Press

1. INTRODUCTION

In 1980, the paper entiled "A Strong Stieltjes Moment Problem" by William B. Jones, W. J. Thron, and Haakon Waadeland appeared and opened up the study of strong distributions and orthogonal Laurent polynomials. Several examples of orthogonal Laurent polynomial are in the literature including [4–6, 9–11, 20]. In [21], several strong distributions were introduced and here we closely examine the strong Chebyshev distribution which first appeared there. Our reasons for developing this example are two-fold. The first is that examples often provide insight that suggests further lines of study. Second, the classical Chebyshev polynomials

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have proved very useful from a numerical point of view and we are optimistic that there is a similar potential for the Chebyshev Laurent polynomials.

The strong Chebyshev distribution, $\psi(x)$, defined in [21] by

$$\psi'(x)=\begin{cases}\dfrac{|x|}{\sqrt{b^2-x^2}\,\sqrt{x^2-a^2}}, & x\in B\equiv[-b,-a]\cup[a,b],\ 0<a<b<\infty,\\ 0, & x\notin B,\end{cases} \tag{1}$$

is a generalization of the classical Chebyshev distribution $\psi_0(x)$ where $\psi_0'(x)=(1-x^2)^{-1/2}$, $x\in[-1,1]$, in the sense that as $a\to 0$ and $b\to 1$, $\psi'(x)\to\psi_0'(x)$. We will show that (1) is a strong symmetric distribution, give closed form expressions for the Chebyshev Laurent polynomials, and establish convergence (to an explicit function) of the continued fraction associated with them.

Before discussing the problem in further detail, we review pertinent terminology and notation, and recall certain results from the literature. A strong distribution function is a bounded, nondecreasing function $\alpha(x)$ with infinitely many points of increase in (a, b) such that its moments

$$\mu_m=\int_a^b x^m\,d\alpha(x),\qquad m=0,\pm 1,\pm 2,\ldots$$

are all finite. Let $\{\mu_n\}_{-\infty}^{\infty}$ be a bisequence of complex numbers, and let $-\infty\leqslant a\leqslant b\leqslant\infty$. The strong moment problem asks whether it is possible to find a strong distribution function $\alpha(x)$ such that

$$\mu_m=\int_\alpha^a x^m\,d\alpha(x),\qquad m=0,\pm 1,\pm 2,\ldots.$$

Two solutions $\alpha(x)$ and $\psi(x)$ are said to be *substantially equal* if there exists a constant C such that $\psi(x)=\alpha(x)+C$ at all common points of continuity. A strong distribution $\alpha(x)$ is said to be *normalized* if $\alpha(-\infty)=0$. If $\alpha(x)$ and $\psi(x)$ are normalized and substantially equal, then $\psi(x)=\alpha(x)$ at all common points of continuity. Since any strong distribution can be normalized by subtracting from it a suitable constant, we restrict our attention to normalized strong distributions. If $\alpha(x)$ is a solution to a strong moment problem, and all other solutions are substantially equal to $\alpha(x)$, then $\alpha(x)$ is the *substantially unique solution* and the moment problem is said to be *determined.* If $\alpha(x)$ is not substantially unique, the problem is said to be *indeterminate.*

A *Laurent polynomial*, or *L-polynomial*, is a rational function of a nonzero, real variable x with the form $R(x)=\sum_{i=m}^{n} r_i x^i$, where $m, n\in\mathbb{Z}$, $m\leqslant n$, and r_i complex for $i=m, \ldots, n$. $R(x)$ is said to be *real* if $r_i \in \mathbb{R}$ for $i=m, \ldots, n$. The set of all Laurent polynomials is denoted by $\mathscr{R}$, while $\mathscr{R}_{m,n}$ represents the set of all Laurent polynomials of the form $R(x)=\sum_{i=m}^{n} r_i x^i$. Such a vector space, $\mathscr{R}_{m,n}$, is called an *L*-space. Two classes of L-polynomials that arise in the study of orthogonal Laurent polynomials are

$$\mathscr{R}_{2m}=\{R\in\mathscr{R}_{-m,m}\colon \text{the coefficient of } x^m \text{ is nonzero}\}$$

and

$$\mathscr{R}_{2m+1}=\{R\in\mathscr{R}_{-(m+1),m}\colon \text{the coefficient of } x^{-(m+1)} \text{ is nonzero}\}$$

for all integers $m\geqslant 0$. For every L-polynomial, $R(x)$, there exists a unique n such that $R(x)\in\mathscr{R}_n$.

Let $\{\mu_n\}_{-\infty}^{\infty}$ be a bisequence of complex numbers and let $\mathscr{L}$ be a complex-valued function defined on the vector space $\mathscr{R}$ by

$$\mathscr{L}[R(x)]=\sum_{i=m}^{n} r_i\mu_i,$$

where $R(x)=\sum_{i=m}^{n} r_i x_i$. The linear functional $\mathscr{L}$ is called the *strong moment functional* determined by the bisequence of moments $\{\mu_n\}_{-\infty}^{\infty}$. $\mathscr{L}$ is said to be *symmetric* if $\mu_{2n+1}=0$ for all $n\in\mathbb{Z}$. A sequence of L-polynomials $\{R_n(x)\}_{n=0}^{\infty}$ is an *orthogonal Laurent polynomial sequence* (OLPS) with respect to $\mathscr{L}$ if $R_k(x)\in\mathscr{R}_k$ for each $k\geqslant 0$ and

$$\mathscr{L}[R_m(x)\,R_n(x)]=K_n\,\delta_{m,n}$$

for all $m, n\geqslant 0$, where $K_n\neq 0$ for all $n\geqslant 0$ and $\delta_{m,n}$ is the Kronecker delta function.

A strong moment functional $\mathscr{L}$ is said to be *positive-definite* if $\mathscr{L}[R(x)]>0$ for all $R(x)\in\mathscr{R}$ such that $R(x)$ is not identically zero and $R(x)\geqslant 0$ for all $x\in\mathbb{R}\backslash\{0\}$. $\mathscr{L}$ is *positive-definite* on $E\subset\mathbb{R}\backslash\{0\}$ if $\mathscr{L}[R(x)]>0$ for all $R(x)\in\mathscr{R}$ such that $R(x)\not\equiv 0$ on E and $R(x)\geqslant 0$ on E. A positive-definite strong moment functional has real moments $\{\mu_n\}_{-\infty}^{\infty}$ [2, 16, 19] and there exists a real OLPS $\{R_n(x)\}_{n=0}^{\infty}$ corresponding to $\mathscr{L}$; see [2, 12, 13, 15, 18]. Furthermore, there exists a strong distribution function $\psi(x)$ such that

$$\mu_m=\mathscr{L}[x^m]=\int_{-\infty}^{\infty} x^m\,d\psi(x), \qquad m=0, \pm 1, \pm 2, \ldots.$$

Such a strong distribution function is called a *representative* of $\mathscr{L}$. If $\alpha(x)$ is a representative and $\psi(x)$ is substantially equal to $\alpha(x)$, then $\psi(x)$ is also a representative of $\mathscr{L}$. If all representatives of $\mathscr{L}$ are substantially equal, then $\mathscr{L}$ is said to be *determinate*. Note that it may be possible for $\mathscr{L}$ to be represented by substantially unequal distribution functions. One such example is the strong moment problem associated with the log-normal distribution [1, 4, 5, 6, 20]. Two substantially unequal distributions are given explicitly in [4], a third representative is given in [20], and two discrete natural representatives are identified in [1]. Thus the log-normal distribution is an example of a distribution corresponding to an *indeterminate* strong moment problem. In contrast, the strong Chebyshev distribution corresponds to a *determined* strong moment problem [8].

In Section 2, we show that the strong Chebyshev distribution represents a symmetric strong moment functional $\mathscr{L}$-positive-definite on $B = [-b, -a] \cup [a, b]$, $0 < a < b < \infty$. Since $\mathscr{L}$ is positive-definite, there exists an associated (monic) OLPS, $\{R_n(x)\}_{n=0}^{\infty}$. In Section 3, we derive the Chebyshev L-polynomials from the classical Chebyshev polynomials using a change of variable, show that they are orthogonal with respect to $\Psi'(x)$, and obtain recurrence relations and closed form expressions for them.

The continued fraction $\mathrm{K}(\alpha_n(z)/\beta_n(z))$ formed from the coefficients of the three-term recurrence relation associated with the OLPS has the form

$$K\left(\frac{\alpha_n(z)}{\beta_n(z)}\right) = \frac{d_1}{z - a_1/z} - \frac{d_2}{z - a_2/z} - \frac{d}{z - a_3/z} - \frac{d_4}{z - a_4/z} - \cdots,$$

where a_n and d_n are constant for $n \geqslant 3$, and hence the tail of the continued fraction is periodic. We use convergence results from [14, 18] to establish uniform convergence on compact subsets of $\mathbb{C} \backslash B$ to a function $F(z)$. Further, an analysis using the relatively simple form of $K(\alpha_n(z)/\beta_n(z))$ allows us to determine $F(z)$ explicitly. Using a result from [22], we then show that a suitably chosen branch of $F(z)$ is equivalent to the Stieltjes transform of the strong Chebyshev distribution. We concude the paper by deriving explicit expressions for the zeros of the Chebyshev Laurent polynomials.

2. THE STRONG CHEBYSHEV DISTRIBUTION

A. Sri Ranga and J. H. McCabe [21] defined the strong Chebyshev distribution by

$$\psi'(x) = \begin{cases} \dfrac{|x|}{\sqrt{(b^2 - x^2)(x^2 - a^2)}}, & x \in B := [-b, -a] \cup [a, b],\ 0 < a < b < \infty, \\ 0, & x \notin B. \end{cases}$$

The strong Chebyshev distribution can be obtained from the classical Chebyshev distribution $\psi_0(x)$ by a quadratic transformation along the lines indicated in [7]. In particular, if

$$T(x) := cx^2 - d, \qquad \text{where} \quad c = \frac{2}{b^2 - a^2} \qquad \text{and} \qquad d = \frac{b^2 + a^2}{b^2 - a^2},$$

then $\psi'(x) = \frac{1}{2}|T'(x)|\, \psi_0'(T(x))$. From the two distributions, it is clear that the transformation must be quadratic. The exact form of the quadratic transformation follows from the equation $\psi'(x) = \frac{1}{2}|T'(x)|\, \psi_0'(T(x))$.

We claim that $\psi(x)$ is indeed a strong distribution, implying that it is a bounded, nondecreasing function with infinitely many points of increase such that its moments are all finite. The fact that $\psi(x)$ is nondecreasing follows immediately from the observation that $\psi'(x) \geqslant 0$ on $\mathbb{R}$. Also, $\psi(x)$ has infinitely many points of increase since it is strictly increasing on B. The fact that $\psi(x)$ is bounded and has finite moments follows from knowledge about μ_0.

We claim that

$$\mu_0 = \int_B d\psi(x) = \int_{-1}^{1} d\psi_0(x) = \pi.$$

Now

$$\begin{aligned}\mu_0 &= \int_B d\psi(x) = \int_B \tfrac{1}{2}|T'(x)|\, \psi_0'(T(x))\, dx \\ &= \int_{-b}^{-a} -\tfrac{1}{2}T'(x)\, \psi_0'(T(x))\, dx + \int_a^b \tfrac{1}{2}T'(x)\, \psi_0'(T(x))\, dx.\end{aligned}$$

By substituting $u = T(x)$ we obtain

$$\begin{aligned}\int_B d\psi(x) &= \int_1^{-1} -\tfrac{1}{2}\psi_0'(u)\, du + \int_{-1}^{1} \tfrac{1}{2}\psi_0'(u)\, du \\ &= \int_{-1}^{1} \psi_0'(u)\, du = \pi.\end{aligned}$$

From this result it follows that all the moment μ_n are finite since

$$\begin{aligned}|\mu_n| &\leqslant \int_B \frac{|x|^{n+1}\, dx}{\sqrt{(b^2 - x^2)(x^2 - a^2)}} \\ &\leqslant \begin{cases} b^n\mu_0 = b^n\pi < \infty & \text{if} \quad n > 0 \\ a^n\mu_0 = a^n\pi < \infty & \text{if} \quad n < 0. \end{cases}\end{aligned}$$

To show that $\psi(x)$ is bounded, note that for any real x,

$$\psi(x) = \int_{-\infty}^{x} d\psi(t),$$

assuming $\psi(x)$ has been normalized so that $\psi(-\infty) = 0$. In particular, $\psi(-b) = 0$, while for $x \in \mathbb{R}$ arbitrary,

$$\psi(x) = \int_{-b}^{x} d\psi(t) \leqslant \int_{-b}^{b} d\psi(t) = \mu_0 = \pi$$

and hence $\psi(x)$ is bounded.

$\psi(x)$ is symmetric since the odd moments vanish as is evident from

$$\mu_{2n+1} = \int_{-b}^{-a} \frac{x^{2n+1}\,|x|\,dx}{\sqrt{(b^2-x^2)(x^2-a^2)}} + \int_{a}^{b} \frac{x^{2n+1}\,|x|\,dx}{\sqrt{(b^2-x^2)(x^2-a^2)}}$$
$$= 0.$$

Thus, $\psi(x)$ is a symmetric strong distribution.

3. CHEBYSHEV LAURENT POLYNOMIALS

In this section, we will introduce a change of variable that will allow us to obtain several results about Chebyshev L-polynomials from results about classical Chebyshev polynomials. Using the transformation we will obtain a sequence of L-polynomials from the Chebyshev polynomials, show that the sequence of L-polynomials is orthogonal with respect to $\psi(x)$, obtain the three-term recurrence formulae, and find closed form expressions for the Chebyshev L-polynomials.

The change of variable which will be used is $y = (1/(b-a))(x - ab/x)$, which results from determining a y such that $\psi_0'(y) = (b-a)\,\psi'(x)$. Let $\{T_n(x)\}_0^\infty$ be the classical monic Chebyshev polynomials and define a sequence of L-polynomials $\{R_n(x)\}_0^\infty$ by

$$R_0(x) = 1$$
$$R_{2n}(x) = (b-a)^n\, 2^{-n+1} T_n(y), \qquad n \geqslant 1 \tag{2}$$
$$R_{2n+1}(x) = \frac{(-1)^n R_{2n}(x)}{(ab)^n x}, \qquad n \geqslant 0.$$

3.1. *Orthogonality with respect to* $\psi'(x)$

Here we will show that $\{R_n(x)\}_0^\infty$ is orthogonal with respect to $\psi(x)$. Recall that for the classical Chebyshev polynomials, $\{T_n(y)\}_0^\infty$, the orthogonality relation is given by

$$\frac{1}{\pi}\int_{-1}^{1} T_n(y)\,T_n(y)\,\frac{dy}{\sqrt{1-y^2}} = \begin{cases} 1 & \text{for} \quad n=m=0 \\ \frac{1}{2}\,\delta_{mn} & \text{otherwise.} \end{cases}$$

We will use the change of variable $y=(1/(b-a))(x-ab/x)$ to show that $\{r_n(x)\}_0^\infty$ is orthogonal with respect to $\psi(x)$, where $r_{2n}(x)=T_n(y)$ and $r_{2n+1}(x)=(1/x)\,r_{2n}(x)$ for $n\geqslant 0$.

First note that the change of variable $y=(1/(b-a))(x-ab/x)$ can also be written as

$$x=\frac{(b-a)\,y\pm\sqrt{(b-a)^2\,y^2+4ab}}{2}.$$

Thus, under this transformation, the interval $[-1, 1]$ for y becomes the union of the two intervals $[-b, -a]$ and $[a, b]$ for x. Also,

$$dy=\frac{1}{b-a}\left(1+\frac{ab}{x^2}\right)dx \qquad \text{and} \qquad \frac{1}{\sqrt{1-y^2}}=\frac{(b-a)\,|x|}{\sqrt{(b^2-x^2)(x^2-a^2)}}.$$

Therefore,

$$\begin{aligned}\frac{1}{\pi}\int_{-1}^{1} & T_n(y)\,T_m(y)\,\frac{dy}{\sqrt{1-y^2}} \\ &=\frac{1}{\pi}\int_B r_{2n}(x)\,r_{2m}(x)\left(1+\frac{ab}{x^2}\right)\frac{|x|\,dx}{\sqrt{(b^2-x^2)(x^2-a^2)}} \\ &=\begin{cases} 1 & \text{for} \quad n=m=0 \\ \frac{1}{2}\,\delta_{mn} & \text{otherwise.} \end{cases}\end{aligned}$$

For $n=m$, it is clear that

$$\int_B r_{2n}^2(x)\,d\psi(x)>0 \qquad \text{and} \qquad \int_B r_{2n+1}^2(x)\,d\psi(x)>0.$$

Assume $n \neq m$. Then,

$$\frac{1}{\pi} \int_B r_{2n}(x)\, r_{2m}(x) \left(1 + \frac{ab}{x^2}\right) d\psi(x) = 0$$

yielding

$$\int_B r_{2n}(x)\, r_{2m}(x)\, d\psi(x) + ab \int_B r_{2n+1}(x)\, r_{2m+1}(x)\, d\psi(x) = 0.$$

We claim that

$$\int_B r_{2n}(x)\, r_{2m}(x)\, d\psi(x) = \int_B r_{2n+1}(x)\, r_{2m+1}(x)\, d\psi(x) = 0.$$

Note that by symmetry

$$\int_B r_{2n}(x)\, r_{2m}(x)\, d\psi(x) = 2 \int_a^b r_{2n}(x)\, r_{2m}(x)\, d\psi(x)$$

and

$$\int_B r_{2n+1}(x)\, r_{2m+1}(x)\, d\psi(x) = 2 \int_a^b r_{2n+1}(x)\, r_{2m+1}(x)\, d\psi(x).$$

Therefore,

$$\int_B r_{2n}(x)\, r_{2m}(x)\, d\psi(x) + ab \int_a^b \frac{1}{x^2}\, r_{2n}(x)\, r_{2n}(x)\, d\psi(x) = 0.$$

Now observe that

$$\begin{aligned} \frac{a}{b} \int_a^b r_{2n}(x)\, r_{2m}(x)\, d\psi(x) &\leqslant ab \int_a^b \frac{1}{x^2}\, r_{2n}(x)\, r_{2m}(x)\, d\psi(x) \\ &\leqslant \frac{b}{a} \int_a^b r_{2n}(x)\, r_{2m}(x)\, d\psi(x). \end{aligned}$$

Hence

$$\left(1 + \frac{a}{b}\right) \int_a^b r_{2n}(x)\, r_{2m}(x)\, d\psi(x) \leqslant 0 \leqslant \left(1 + \frac{b}{a}\right) \int_a^b r_{2n}(x)\, r_{2m}(x)\, d\psi(x)$$

from which it follows that

$$\int_a^b r_{2n}(x)\, r_{2m}(x)\, d\psi(x)=0$$

and hence

$$\int_B r_{2n}(x)\, r_{2m}(x)\, d\psi(x)=0$$

which in turn implies that $\int_B r_{2n+1}(x)\, r_{2m+1}(x)\, d\psi(x)=0$. Since $R_{2n}(x)=(b-a)^n\, 2^{-n+1} r_{2n}(x)$ and $R_{2n+1}(x)=((-1)^n\,(b-a)^n\, 2^{-n+1}/(ab)^n)\, r_{2n+1}(x)$, the L-polynomial sequence $\{R_n(x)\}_0^\infty$ is orthogonal with respect to $\psi(x)$.

3.2. *Three-Term Recurrence Relations and Closed Form Expressions*

The three-term recurrence relations for the $\{R_n(x)\}_0^\infty$ will be derived from the three-term recurrence relations for $\{T_n(x)\}_0^\infty$. Since $\{T_n(x)\}_0^\infty$ are the Chebyshev polynomials of the first kind,

$$T_0(x)=1, \qquad T_1(x)=1$$

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x), \qquad n\geqslant 1.$$

Thus,

$$R_0(x)=1, \qquad R_2(x)=x-\frac{ab}{x} \tag{3}$$

and

$$\begin{aligned}(b-a)^{n+1}\, 2^{-n}T_{n+1}(y)&=(b-a)\, y(b-a)^n\, 2^{-n+1}T_n(y)\\&\quad-(b-a)^2\, 2^{-2}(b-a)^{n-1}\, 2^{-n+2}T_{n+1}(y)\end{aligned}$$

yielding

$$R_{2n+2}(x)=\left(x-\frac{ab}{x}\right) R_{2n}(x)-\frac{(b-a)^2}{4} R_{2n-2}(x) \qquad \text{for} \quad n\geqslant 1. \tag{4}$$

Relation (2) will now be used to derive closed form expressions for $R_{2n}(x)$. Starting with

$$T_n(y)=\frac{n}{2}\sum_{j=0}^{[n/2]} \frac{(-1)^j\,(n-j-1)!}{j!\,(n-2j)!}(2y)^{n-2j} \qquad \text{and} \qquad y=\frac{1}{b-a}(x-ab/x),$$

we have

$$R_{2n}(x)=(b-a)^n\,2^{-n+1}\,\frac{n}{2}\sum_{j=0}^{[n/2]}\frac{(-1)^j\,(n-j-1)!}{j!\,(n-2j)!}\left(\frac{2(x-ab/x)}{b-a}\right)^{n-2j}$$

$$=n\sum_{j=0}^{[n/2]}\frac{(-1)^j\,(n-j-1)!}{j!\,(n-2j)!}\left(\frac{b-a}{2}\right)^{2j}\left(x-\frac{ab}{x}\right)^{n-2j}$$

$$=n(ab)^n\sum_{j=0}^{[n/2]}\frac{(-1)^{n+j}\,(n-j-1)!}{j!}$$

$$\times\left(\frac{b-a}{2ab}\right)^{2j}\sum_{k=0}^{n-2j}\frac{(-1)^k\,x^{2k+2j-n}}{k!\,(n-2j-k)!\,(ab)^k}$$

$$=n(ab)^n\sum_{j=0}^{[n/2]}\frac{(-1)^{n-j}\,(n-j-1)!}{j!}\left(\frac{b-a}{2ab}\right)^{2j}$$

$$\times\left[\frac{x^{2j-n}}{(n-2j)!}-\frac{x^{2+2j-n}}{(n-2j-1)!\,(ab)}+\frac{x^{4+2j-n}}{2!\,(n-2j-2)!\,(ab)^2}\right.$$

$$-\frac{x^{6+2j-n}}{3!\,(n-2j-3)!\,(ab)^3}+\cdots+\frac{(-1)^{n-1}\,x^{n-2j-2}}{(n-2j-1)!\,(ab)^{n-2j-1}}$$

$$\left.+\frac{(-1)^n\,x^{n-2j}}{(n-2j)!\,(ab)^{n-2j}}\right].$$

Therefore, $R_{2n}(x)=\sum_{k=-n}^{n} r_{2n,\,k}\,x^k$ where

$$r_{2n,\,-n+2j}$$

$$=\frac{(-1)^{n-j}\,n(ab)^{n-2j}\,(b-a)^{2j}}{2^{2j}}\sum_{k=0}^{j}\frac{(n-j-1+k)!\,2^{2k}(ab)^k}{k!\,(j-k)!\,(n-2j+k)!\,(b-a)^{2k}}$$

for $j=0, 1, ..., [n/2]$ and

$$r_{2n,\,n-2j}=(-1)^j\,n\,\frac{(b-a)^{2j}}{2^{2j}}\sum_{k=0}^{j}\frac{(n-j-1+k)!\,(ab)^k\,2^{2k}}{k!\,(j-k)!\,(n-2j+k)!\,(b-a)^{2k}}$$

for $j=0, 1, ..., [n/2]$.

Notice that $r_{2n,\,-n+2j}=(-1)^n\,(ab)^{n-2j}\,r_{2n,\,n-2j}$ for $j=0, 1, ..., [n/2]$.

4. THE ASSOCIATED CONTINUED FRACTION

This section is devoted to the continued fraction associated with the three-term recurrence relations given in (3) and (4). We begin by reviewing some basic terminology related to a continued fraction of the form

$$b_0(z) + K_{n=1}^{\infty}\left(\frac{a_n(z)}{b_n(z)}\right). \tag{5}$$

The functions $a_n(z)$ and $b_n(z)$ are *elements* of the continued fraction and $f_n(z) = b_0(z) + K_{j=1}^{n}\,(a_j(z)/b_j(z))$ is the nth *approximant* of (5). The continued fractions $K_{n=p}^{\infty}(a_n(z)/b_n(z))$, $p = 1, 2, \ldots$, are the *tails* of (5).

The continued fraction (5) is said to *converge* to a function $f(z)$ for $z \in D$ if $(f_n(z))$ converges (pointwise) to $f(z)$ for $z \in D$. It is said to *converge uniformly* to $f(z)$ on D if $(f_n(z))$ converges uniformly to f on D.

A continued fraction $K(a_n/b_n)$ is said to be *periodic with period k* if its elements satisfy the conditions $a_{nk+p} = a_p$, $b_{nk+p} = b_p$, $p = 1, 2, \ldots, k$, $n = 0, 1, 2, \ldots$. A continued fraction $K(a_n/b_n)$ is said to be *limit periodic with period k*, or limit k-periodic, if its elements satisfy $\lim_{n\to\infty} a_{nk+p} = a_p^*$, $\lim_{n\to\infty} b_{nk+p} = b_p^*$, $p = 1, 2, \ldots, k$, when these limits all exist in $\mathbb{C}$. In this case, the elements are said to be limit k-periodic.

The continued fraction arising from the three-term recurrence relations given in Eqs. (3) and (4) is

$$\frac{\pi\,\dfrac{z}{z^2-\gamma}}{1} - \frac{\dfrac{1}{2}\lambda^2\,\dfrac{z^2}{(z^2-\gamma)^2}}{1} + K\left(\frac{-\left[\dfrac{1}{2}\lambda\,\dfrac{z}{z^2-\gamma}\right]^2}{1}\right). \tag{6}$$

We will show that this limit 1-periodic continued fraction converges pointwise on $\mathbb{C}\backslash B$, $B = [-b, -a] \cup [a, b]$, to a function $F(z)$. We then determine an explicit expression for the limit function $F(z)$, and show that a suitably chosen branch of $F(z)$ is equivalent to the Stieltjes transform of the strong Chebyshev distribution. Finally, we show that the continued fraction converges uniformly on compact subsets of $\mathbb{C}\backslash B$. The uniform convergence is shown using results from [18].

4.1. *The Limit Function and Its Analyticity*

In this section, we first determine the pointwise convergence region for the continued fraction (6). This is done with the help of Theorem 3.2 form [14]. We then examine the limit function $F(z)$, and select a branch of $F(z)$ which is analytic on $\mathbb{C}\backslash B$.

THEOREM 4.1. *Let $\gamma = ab$ and $\lambda = b - a$. Then the continued fraction*

$$\frac{\pi \dfrac{z}{z^2-\gamma}}{1} - \frac{\dfrac{1}{2}\lambda^2 \dfrac{z^2}{(z^2-\gamma)^2}}{1} + K\left(\frac{-\left[\dfrac{1}{2}\lambda \dfrac{z}{z^2-\gamma}\right]^2}{1}\right) \tag{7}$$

converges pointwise on $\mathbb{C}\backslash B$ and diverges on B.

Proof. Let $w = z^2$, $w = re^{i\theta}$. Then, as mentioned above, by Theorem 3.2 in [14], the continued fraction (7) converges for all nonzero complex $-\frac{1}{4}\lambda^2 w/(w-\gamma)^2$ unless $w/(w-\gamma)^2$ is real and $w/(w-\gamma)^2 > 1/\lambda^2$, or equivalently, when

$$\frac{r}{(r^2+\gamma^2)\cos\theta - 2\gamma r + i(r^2-\gamma^2)\sin\theta} > \frac{1}{\lambda^2}. \tag{8}$$

In order for expression (8) to be real, it is necessary to have $(r^2-\gamma^2)\sin\theta = 0$, implying $\sin\theta = 0$ or $r = \gamma$. When $\theta = 0$, (8) reduces to $(r-a^2)(r-b^2) < 0$, which holds if and only if $a^2 < r < b^2$. Both $\theta = \pi$ and $r = \gamma$ lead to contradictions. Thus $w/(w-\gamma)^2 > 1/\lambda^2$ if and only if $a^2 < r < b^2$ and $\theta = 0$ and hence the continued fraction diverges if and only $z \in B = [-b, -a] \cup [a, b]$. ∎

Next, we focus on the limit function $F(z)$. We will determine an explicit expression for $F(z)$, find a branch of $F(z)$ that is analytic on $\mathbb{C}\backslash B$, and show that this branch is the Stieltjes transform of the strong Chebyshev distribution.

THEOREM 4.2. *Let $\gamma = ab$ and $\lambda = b - a$. Then the continued fraction*

$$\frac{\pi \dfrac{z}{z^2-\gamma}}{1} - \frac{\dfrac{1}{2}\lambda^2 \dfrac{z^2}{(z^2-\gamma)^2}}{1} + K\left(\frac{-\left[\dfrac{1}{2}\lambda \dfrac{z}{z^2-\gamma}\right]^2}{1}\right) \tag{9}$$

converges on $\mathbb{C}\backslash B$ to $F(z)$, where

$$F(z) = \frac{x}{\sqrt{z^2-b^2}\,\sqrt{z^2-a^2}}. \tag{10}$$

Proof. Let $z \in \mathbb{C}\backslash B$, so that by Theorem 4.1, the continued fraction (9) converges to $F(z)$. Then

$$F(z) = \frac{\pi z}{z^2-\gamma} - \frac{(1/2)\,\lambda^2 z^2}{z^2-\gamma} - f(z) \tag{11}$$

where

$$f(z) = \frac{(1/4)\,\lambda^2 z^2}{z^2 - \gamma - f(z)}.$$

Recalling that $\gamma = ab$ and $\lambda = b - a$, we solve for $f(z)$ obtaining

$$f(z) = \frac{(z^2 - ab) \pm \sqrt{(z^2 - b^2)(z^2 - a^2)}}{2}. \tag{12}$$

Using this expression for $f(z)$ in Eq. (11), and after straightforward simpflications, $F(z)$ becomes

$$F(z) = \mp \frac{\pi z}{\sqrt{z^2 - b^2}\,\sqrt{z^2 - a^2}}. \tag{13}$$

Now we determine whether to choose the positive or negative sign in Eq. (13) for $F(z)$. Let $z = i \in \mathbb{C} \backslash B$. Then the continued fraction (9) converges. Using Theorem 3.2 from [14], the tail $K([\frac{1}{2}\lambda/(1+\gamma)]^2/1)$ of $F(i)$ converges to $-\frac{1}{2} + \frac{1}{2}\sqrt{(\lambda/(1+\gamma))^2 + 1}$. With this expression for the tail of $F(i)$, (9) reduces to

$$F(i) = \frac{-\pi i}{(1+\gamma)(1+P)}, \tag{14}$$

where $P = [\lambda/(1+\gamma)]^2\,[1 + \sqrt{(\lambda/(1+\gamma))^2 + 1}\,]^{-1} > 0$. Also, from (13),

$$F(i) = \pm \frac{\pi i}{\sqrt{1 + b^2}\,\sqrt{1 + a^2}}. \tag{15}$$

Since expressions (14) and (15) for $F(i)$ must be equal, we choose the positive sign in (13) for $F(z)$, which completes the proof. ▮

Using the result (10) of the above theorem, we will show that $F(z)$ is equivalent to the Stieltjes transform of the strong Chebyshev distribution. The Stieltjes transfrom of $\psi(x)$ is

$$S(\psi, z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\psi(x)}{z - x}, \qquad z \in \mathbb{C} \backslash \mathbb{R}.$$

The next theorem is an adaptation of a lemma appearing in [22]. We give this result here, and then use it in proving that $F(z)$ is equal to the Stieltjes transform of the strong Chebyshev distribution.

THEOREM 4.3. *Let $\psi(x)$ be the strong Chebyshev distribution defined in* (1), *with support* $B = [-b, -a] \cup [a, b]$, $0 < a < b < \infty$. *The Stieltjes transform of $\psi(x)$ is*

$$S(\psi, z) = \frac{\pi z}{\sqrt{z^2 - b^2}\,\sqrt{z^2 - a^2}}, \qquad z \in \mathbb{C} \setminus B,$$

provided the roots $\sqrt{z^2 - b^2}$ and $\sqrt{z^2 - a^2}$ are chosen such that $z^2 / \sqrt{z^2 - b^2}\,\sqrt{z^2 - a^2}$ is analytic on $\mathbb{C} \setminus B$ and tends to 1 as z tends to infinity.

THEOREM 4.4. *Let $F(z)$ denote the limit function* (10) *of the continued fraction* (9). *Then a branch of $F(z)$ analytic on $\mathbb{C} \setminus B$ may be chosen such that*

$$F(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\psi(x)}{z - x}, \qquad z \in \mathbb{C} \setminus B. \tag{16}$$

Thus the continued fraction (9) *converges to the Stieltjes transform of the strong Chebyshev distribution.*

Proof. A branch of $F(z)$ analytic on $\mathbb{C} \setminus B$ can easily be derived as follows. Let $r(z) = \sqrt{z^2 - b^2}\,\sqrt{z^2 - a^2}$. A branch of $r(z)$ analytic on $\mathbb{C} \setminus B$ is given by

$$r(z) = \sqrt{r_1 r_2 r_3 r_4}\, e^{i(\theta_1 + \theta_2 + \theta_3 + \theta_4)/2}$$

with $r_1 = |z + b|$, $r_2 = |z + a|$, $r_3 = |z - a|$, $r_4 = |z - b|$, $r_1 + r_2 > b - a$, $r_3 + r_4 > b - a$, $\theta_1 = \arg(z + b)$, $\theta_2 = \arg(z + a)$, $\theta_3 = \arg(z - a)$, $\theta_4 = \arg(z - b)$, $0 \leqslant \theta_k < 2\pi$, $k = 1, 2, 3, 4$ [22]. Furthermore, rewriting $zF(z)$ as

$$zF(z) = \frac{\pi}{\sqrt{1 - b^2/z^2}\,\sqrt{1 - a^2/z^2}},$$

it follows immediately that $\lim_{z \to \infty} zF(z) = \pi$. Therefore, by Theorem 4.3, $F(z)$ is equal to the Stieltjes transform of the strong Chebyshev distribution. ∎

4.2. *Uniform Convergence*

From the results of the previous section, the continued fraction (6) converges on $\mathbb{C} \setminus B$ to the limit function $F(z)$, where $F(z)$ is given in (10). Here, we show that the continued fraction converges uniformly to $F(z)$ on compact subsets of $\mathbb{C} \setminus B$. This is done by applying results given in [18].

Recall that our continued fraction (6) is limit 1-periodic with

$$a_1^*(z) = -\frac{1}{4}\lambda^2 \frac{z^2}{(z^2-\gamma)^2} \tag{17}$$

and $b_1^* = 1$, where $\gamma = ab$ and $\lambda = b - a$. The convergence of (6) is determined by the linear fractional transformation $T(w) = a_1^*(z)/(1+w)$ and by the fixed points $u(z)$ and $v(z)$ of $T(w)$. These fixed points, $u(z)$ and $v(z)$, are $-\frac{1}{2} \pm \frac{1}{2}\sqrt{(z^2-b^2)(z^2-a^2)}/(z^2-ab)$. $T(w)$ is said to be *loxodromic* since $a_1^*(z) \neq 0$ when $z \neq 0$ and $b_1^* = 1 \neq 0$ when $a_1^*(z) = 0$.

The following theorem, adapted to our setting, appears as Theorem 31, Chap. 3 of [18].

THEOREM 4.5. *Let $K(a_n(z)/b_n(z))$ be the limit 1-periodic continued fraction*

$$\frac{\pi \dfrac{z}{z^2-\gamma}}{1} - \frac{\dfrac{1}{2}\lambda^2 \dfrac{z^2}{(z^2-\gamma)^2}}{1} + K\left(\frac{-\left[\dfrac{1}{2}\lambda \dfrac{z}{z^2-\gamma}\right]^2}{1}\right). \tag{18}$$

Let $a_1^*(z)$ and $T(w)$ be as given above, and let $u(z)$, $v(z)$ be the fixed points of $T(w)$. If $D \subset \mathbb{C}$ is an open set where $u(z)$ and $v(z)$ are both finite, then $K(a_n(z)/b_n(z))$ converges uniformly on compact subsets C of D to a function $F(z)$, provided $F(z)$ is finite on C.

We use the result of this theorem in the proof of Theorem 4.6 below.

THEOREM 4.6. *Let $\gamma = ab$ and $\lambda = b - a$. Then the continued fraction*

$$\frac{\pi \dfrac{z}{z^2-\gamma}}{1} - \frac{\dfrac{1}{2}\lambda^2 \dfrac{z^2}{(z^2-\gamma)^2}}{1} + K\left(\frac{-\left[\dfrac{1}{2}\lambda \dfrac{z}{z^2-\gamma}\right]^2}{1}\right) \tag{19}$$

converges uniformly on compact subsets of $\mathbb{C} \backslash B$.

Proof. We see that $u(z)$ and $v(z)$ are both finite on $D = \mathbb{C} \backslash B$. Further, by (10), the limit function $F(z)$ is finite-valued on $\mathbb{C} \backslash B$. Hence, by Theorem 4.5, the continued fraction (19) converges uniformly on compact subsets of $\mathbb{C} \backslash B$. ∎

4.3. *Zeros of the Orthogonal* L-*Polynomials*

We conclude the paper by finding explicit expressions for the zeros of $R_{2n}(x)$. First recall that the zeros of the classical Chebyshev polynomials, $T_n(x)$, are given by

$$x_{n,k} = \cos\left(\frac{2k-1}{n}\pi\right), \qquad k = 1, 2, ..., n. \tag{20}$$

These zeros will be used to find explicit expressions for the zeros of $R_{2n}(x)$. From (2) we see that if $\{z_{2n,k} : k = \pm 1, ..., \pm n\}$ are the zeros of $R_{2n}(x)$ then

$$T_n\left(\frac{z_{n,k} - ab/z_{n,k}}{b-a}\right) = 0, \qquad k = \pm 1, \pm 2, ..., \pm n.$$

Hence for each $k \in \{\pm 1, \pm 2, ..., \pm n\}$ there exists an m_k, $1 \leqslant m_k \leqslant n$, such that

$$z_{2n,k} = \frac{1}{2}(b-a)\cos\left(\frac{2m_k - 1}{2n}\pi\right) \pm \frac{1}{2}\sqrt{(b-a)^2\cos^2\left(\frac{2m_k-1}{2n}\pi\right) + 4ab}.$$

An alternate way of obtaining expressions for the zeros of $R_{2n}(x)$ is to use the fact that $R_{2n}(x)$ is the nth denominator of the continued fraction (19). It can be shown that the fixed points $u(x)$ and $v(x)$ of the linear fractional transformation

$$s(w) = \frac{-(1/4)\lambda^2}{x - ab/x + w}$$

satisfy

$$R_{2n}(x) = [-u(x)]^n + [-v(x)]^n \tag{21}$$

for $n = 3, 4, ...$. Setting $R_{2n}(x) = 0$ and taking the nth root of both sides of (21), we find the zeros of $R_{2n}(x)$ to be

$$\pm\frac{1}{2}(b-a)\cos\left(\frac{2k-1}{2n}\pi\right) \pm \frac{1}{2}\sqrt{(b-a)^2\cos^2\left(\frac{2k-1}{2n}\pi\right) + 4ab}$$

for $k = 1, 2, ..., n$. After simplifying and removing the redundancy, it was shown in [3] that the zeros of $R_{2n}(x)$ are given by $\{\pm t_{2n,k}\}_{k=1}^{n}$ where

$$t_{2n,k} = -\frac{1}{2}(b-a)\cos\left(\frac{2k-1}{2n}\pi\right) + \frac{1}{2}\sqrt{(b-a)^2\cos^2\left(\frac{2k-1}{2n}\pi\right) + 4ab}.$$

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